A **graph** comprises:

a finite non-empty set V of **vertices**;

a finite set E of **edges**;

an end-point function ∂ such that, for each $e \in E$, $\partial(e)$ is the set of vertices which e joins. Thus, for each $e \in E$, the set $\partial(e)$ contains one or two vertices.

Formally, ∂ is a function $E \to \mathbb{P}(V)$ such that, for each $e \in E$, $\#(\partial(e)) = 1$ or 2, where $\mathbb{P}(V)$ is the power set of V (the set of all subsest of V) and $\#(\partial(e))$ denotes the number of elements in the set $\partial(e)$.

Alternatively, to avoid having to define ∂, we could define the edges by their endpoints so that E is a list (possibly with repeats) of one or two element sets of vertices. This is *Wilson's* approach, although he calls E a 'family'; mathematically, E is a **bag**. The disadvantage of this approach is that it does not allow us to identify different edges joining the same vertices.

If $\partial(e) = \{v, w\}$ then e **joins** v and w; if $\partial(e) = \{v\}$ then e is a **loop**.

If $\partial(e_1) = \partial(e_2)$ then e_1 and e_2 are **multiple edges.**

If $\partial(e) = \{v_1, v_2\}$ then v_1 and v_2 are **incident** with e, and e is **incident** with v_1 and v_2.

The vertices v_1 and v_2 are **adjacent** if there exists an edge e such that $\partial(e) = \{v_1, v_2\}$.

An **isolated vertex** is a vertex which has no edges incident with it.

A **simple graph** is a graph which has no loops or multiple edges.

The **degree** of a vertex is the number of (distinct) edges incident with it. A loop contribute 2 to the degree of the vertex with which it is incident.

The **degree sequence** of a graph G is the sequence of its vertex degrees, written in non-decreasing order.

The **adjacency matrix** of a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ where a_{ij} is the number of edges which join v_i and v_j.

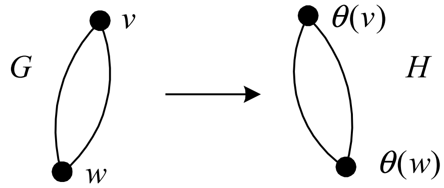
A graph is **regular** of **degree *r*** if every vertex has degree r.

A graph is **complete** if it has no loops and every pair of distinct vertices is joined by a unique edge. The complete graph with n vertices is denoted K_n.

A graph G is **bipartite** if the vertex set V is the union of two disjoint, non-empty sets V_1 and V_2 such that each edge of G joins and edge of V_1 and an element of V_2. (The sets V_1 and V_2 form a partition of the vertex set V.)

A **complete bipartite** graph is a bipartite graph in which each vertex in V_1 is joined to each vertex in V_2 by a unique edge. If V_1 has r vertices and V_2 has s vertices (symbolically, $\#(V_1) = r$ and $\#(V_2) = s$) then the corresponding complete bipartite graph is denoted $K_{r,s}$.

Let G and H be graphs. An **isomorphism** $G \to H$ is a bijection θ: $V(G) \to V(H)$ such that, for all $v_1, v_2 \in V(G)$, the number of edges joining v_1 and v_2 is the same as the number of edges joining $\theta(v_1)$ and $\theta(v_2)$ in H. If there exists an isomorphism $G \to H$ then G and H are **isomorphic**, written $G \cong H$.



Isomorphism Principle

Let G and H be two graphs.

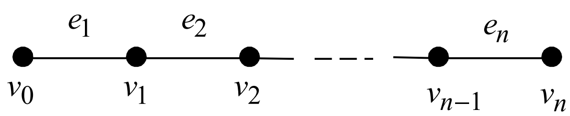
To show that G is isomorphic to H, we must find an appropriate isomorphism $G \to H$.

To show that G is not isomorphic to H, we must find a graph-theoretic property which one graph has but the other does not.

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and every edge of H is also an edge of H. We write $H \leq G$ to mean H is a subgraph of G.

A **walk** of **length** $\boldsymbol{n}$ is a finite sequence of edges $e_1, e_2, \ldots, e_n$ where, for each $i = 1, 2, \ldots, n$, $\partial(e_i) = \{v_{i-1}, v_i\}$ (where $v_{i-1} = v_i$ is allowed); in other words, each successive pair of edges in the sequence is adjacent to a common vertex.

The **associated vertex sequence** is $v_0, v_1, v_2, \ldots, v_n$; the vertex v_0 is the **initial vertex** and the vertex v_n is the **final vertex** of the walk.



A **trail** is a walk in which all the edges are distinct.
A **closed** walk or trail starts and ends at the same vertex; that is $v_0 = v_n$.
A **path** is a trail in which all vertices are distinct (except, possibly, the first and last vertex).
A **cycle** (or **circuit**) is a closed path.

A graph is **connected** if, for each pair of distinct vertices, there is a path from one to the other; a graph which is not connected is **disconnected**.

A graph splits into a number of connecetd pieces called components; a connected graph has one component. Formally, a **component** of G is a maximal connected subgraph of G.