Graphs

A **graph** comprises:

a finite non-empty set V of **vertices**;

a finite set E of **edges**;

an end-point function ∂ such that, for each $e \in E$, $\partial(e)$ is the set of vertices which e joins. Thus, for each $e \in E$, the set $\partial(e)$ contains one or two vertices.

Formally, ∂ is a function $E \to \mathbb{P}(V)$ such that, for each $e \in E$, $\#(\partial(e)) = 1$ or 2, where $\mathbb{P}(V)$ is the power set of V (the set of all subsest of V) and $\#(\partial(e))$ denotes the number of elements in the set $\partial(e)$.

If $\partial(e) = \{v, w\}$ then e **joins** v and w; if $\partial(e) = \{v\}$ then e is a **loop**.

If $\partial(e_1) = \partial(e_2)$ then e_1 and e_2 are **multiple edges.**

If $\partial(e) = \{v_1, v_2\}$ then v_1 and v_2 are **incident** with e, and e is **incident** with v_1 and v_2.

The vertices v_1 and v_2 are **adjacent** if there exists an edge e such that $\partial(e) = \{v_1, v_2\}$.

An **isolated vertex** is a vertex which has no edges incident with it.

A **simple graph** is a graph which has no loops or multiple edges.

The **degree** of a vertex is the number of (distinct) edges incident with it. A loop contribute 2 to the degree of the vertex with which it is incident.

The **degree sequence** of a graph G is the sequence of its vertex degrees, and may be written in non-increasing or non-decreasing order.

The **adjacency matrix** of a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ where a_{ij} is the number of edges which join v_i and v_j.

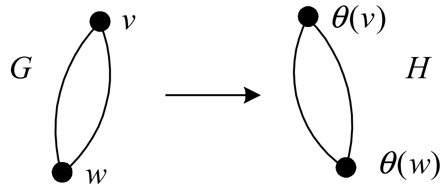
A graph is **regular** of **degree *r*** if every vertex has degree r.

A graph is **complete** if it has no loops and every pair of distinct vertices is joined by a unique edge. The complete graph with n vertices is denoted K_n.

A graph G is **bipartite** if the vertex set V is the union of two disjoint, non-empty sets V_1 and V_2 such that each edge of G joins and edge of V_1 and an element of V_2. (The sets V_1 and V_2 form a partition of the vertex set V.)

A **complete bipartite** graph is a bipartite graph in which each vertex in V_1 is joined to each vertex in V_2 by a unique edge. If V_1 has r vertices and V_2 has s vertices (symbolically, $\#(V_1) = r$ and $\#(V_2) = s$) then the corresponding complete bipartite graph is denoted $K_{r,s}$.

Let G and H be graphs. An **isomorphism** $G \to H$ is a bijection θ: $V(G) \to V(H)$ such that, for all $v_1, v_2 \in V(G)$, the number of edges joining v_1 and v_2 is the same as the number of edges joining $\theta(v_1)$ and $\theta(v_2)$ in H. If there exists an isomorphism $G \to H$ then G and H are **isomorphic**, written $G \cong H$.



Isomorphism Principle

Let G and H be two graphs.

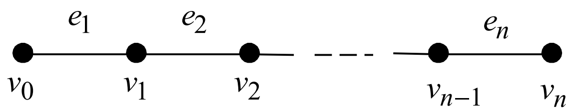
To show that G is isomorphic to H, we must find an appropriate isomorphism $G \to H$.

To show that G is not isomorphic to H, we must find a graph-theoretic property which one graph has but the other does not.

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and every edge of H is also an edge of G. We write $H \leq G$ to mean H is a subgraph of G.

A **walk** of **length** $\boldsymbol{n}$ is a finite sequence of edges $e_1, e_2, \ldots, e_n$ where, for each $i = 1, 2, \ldots, n$, $\partial(e_i) = \{v_{i-1}, v_i\}$ (where $v_{i-1} = v_i$ is allowed); in other words, each successive pair of edges in the sequence is adjacent to a common vertex.

The **associated vertex sequence** is $v_0, v_1, v_2, \ldots, v_n$; the vertex v_0 is the **initial vertex** and the vertex v_n is the **final vertex** of the walk.



A **trail** is a walk in which all the edges are distinct.

A **closed** walk or trail starts and ends at the same vertex; that is $v_0 = v_n$.

A **path** is a trail in which all vertices are distinct (except, possibly, the first and last vertex).

A **cycle** (or **circuit**) is a closed path.

A graph is **connected** if, for each pair of distinct vertices, there is a path from one to the other; a graph which is not connected is **disconnected**.

A graph splits into a number of connecetd pieces called components; a connected graph has one component. Formally, a **component** of G is a maximal connected subgraph of G.

An **Eulerian trail** in a graph G is a closed trail which includes every edge of G. A graph is **Eulerian** if it has an Eulerian trail. A graph is **semi-Eulerian** if there is an open trail which includes every edge of G.

A **Hamiltonian cycle** in a graph G is a cycle which passes through every vertex of G. A graph is **Hamiltonian** if it has a Hamiltonian cycle.

Digraphs

A **directed graph** or **digraph** D comprises:

- a finite non-empty set $V = V(D)$ of vertices,
- a finite set $A = A(D)$ of arcs, and
- an end-point function $\partial : A \to V \times V$ such that, for every arc a, $\partial(a)$ is the ordered pair of vertices which a joins. (Recall that $V \times V$ is the set of all ordered pairs of elements of V.)

If $\partial(a) = (v_1, v_2)$ we say that a is an arc **from** v_1 **to** v_2 and that v_1 is the **initial vertex** of a, v_2 is the **final vertex** of a.

Given $v \in V$, the number of distinct arcs with final vertex v is the **in-degree** of v, denoted indeg(v) and number of distinct arcs with initial vertex v is the **out-degree** of v, denoted outdeg(v).

Special types of digraph

D is **simple** if it has no loops or multiple arcs.

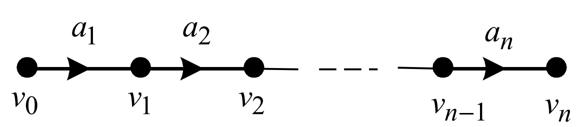
D is **regular of degree** (r, s) if every vertex has in-degree r and out-degree s.

D is a **null digraph** if it has no arcs.

Paths and cycles

In a digraph, walks (hence trails, paths and cycles) are directed.

A **walk** in D is a sequence of arcs $a_1, a_2, \ldots, a_n$ such that, for $i = 1, \ldots, n-1$, the final vertex of a_i is the same as the initial vertex of a_{i+1}. The walk defines an **associated vertex sequence** $v_0, v_1, v_2, \ldots, v_n$ where, for $i = 1, \ldots, n-1$, $\partial(a_i) = (v_{i-1}, v_i)$. We say the walk is **from** v_0 **to** v_n.



Connectivity

A digraph D is **connected**, or **weakly connected**, if its underlying graph is connected.

A digraph D is **strongly connected** if, for every pair of distinct vertices v and w, there exists a path in D from v to w.

Trees

A **tree** is a connected graph which contains no cycles; a **forest** is a graph which contains no cycles.

Thus a connected forest is a tree; each component of a forest is a tree.

Theorem Let T be a graph with n vertices. The following statements are equivalent (so that, if any one statement is true for T then all the statements are true for T and, conversely, if any one statement is false for T then all the statements are false for T).

(i) T is a tree.

(ii) T is connected and has $n - 1$ edges.

(iii) T is connected and every edge is a bridge.

(iv) Given any pair of distinct vertices in T, there is a unique path joining them.

(v) T contains no cycles, but adding any additional edge creates a cycle.

Let G be a connected graph. A **spanning tree** in G is a subgraph T which is a tree and which contains every vertex of G.

Networks

Flows and Cuts

Let N be a network with source S and sink T.

(i) A **cut** of N is a set of arcs which, if removed from the network, produces a digraph with two components, X containing the source S and Y containing the sink T.

(ii) The **capacity** of a cut is the sum of the capacities of those arcs in the cut which are directed from X to Y,

(iii) A cut is **minimum** if its capacity is less than or equal to the capacity of any other cut.

Planarity and Colouring

A **plane graph** is a graph in which no two edges intersect. A **planar graph** is isomorphic to a plane graph.

Theorem K_5 is non-planar and $K_{3,3}$ is non-planar.

Two graphs are **homeomorphic** if both can be derived from the same graph by inserting new vertices of degree 2 into its edges.

Theorem (Kuratowski) A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem (Euler's Formula) Let G be a connected plane graph with n vertices, m edges and f faces. Then

$$n - m + f = 2.$$

Corollary If G is a connected simple plane graph with n (≥ 3) vertices and m edges, then

$$m \leq 3n - 6.$$

Theorem (Euler's Formula for disconnected graphs) If G is a plane graph with n vertices, m edges, f faces and k components, then

$$n - m + f = k + 1.$$

Graphs on other surfaces

A surface is said to be of **genus** g if it is topologically equivalent to a sphere with g handles (or a doughnut with g holes).

A graph which can be drawn without crossings on a surface of genus g, but not on one of genus $g - 1$, is called a **graph of genus g**.

Theorem Let G be a connected graph of genus g, with n vertices, m edges and f faces. Then

$$n - m + f = 2 - 2g.$$

Dual Graphs

Theorem Let G be a plane connected graph with n vertices, m edges and f faces, and let its dual G^* have n^* vertices, m^* edges and f^* faces. Then $n^* = f$, $m^* = m$, $f^* = n$.

Theorem Let G be a plane connected graph. Then G^{**} is isomorphic to G.

Graph colouring

A graph G (without loops) is ***k*-colourable** if we can colour the vertices with k colours so that no two adjacent vertices have the same colour. If G is k-colourable but not $(k-1)$-colourable, then G is ***k*-chromatic** or G has **chromatic number** k.

Theorem If G is a simple graph with largest vertex-degree p, then G is $(p+1)$-colourable.

Theorem (Brooks) If G s a simple connected graph (not K_n) with largest vertex-degree p (≥ 3), then G is p-colourable.

Theorem (4-colour theorem) Every planar graph is 4-colourable.

The **chromatic polynomial** $P_G(k)$ of a simple graph G is the number of ways of colouring the vertices of G with k colours so that no two adjacent vertices have the same colour.

Theorem Let G be a simple graph. Let G_1 and G_2 be graphs obtained from G by deleting and contracting (respectively) an edge e. Then

$$P_G(k) = P_{G_1}(k) - P_{G_2}(k)$$

Properties of $P_G(k)$

1 $P_G(k)$ is a polynomial.

2 If G is a null graph on n vertices, then $P_G(k) = k^n$.

3 If G has n vertices, then $P_G(k)$ has degree n.

4 The coefficient of k^n is one.

5 The coefficient of k^{n-1} is $-m$ (where m is the number of edges in G).

Petri-nets

A **Petri net** is a directed bipartite graph in which the two classes of vertices are called **places** and **transitions**. Places are drawn as circles and transitions as bars.

A **marking** of a Petri net assigns each place a natural number. If n is assigned to place p, we say that there are n **tokens** on p. The tokens are represented as black dots.

Place p is an **input place** for transition t if there is an edge directed from p to t. An **output place** is similarly defined.

If every input place for a transition t has at least one token, then t is **enabled**.

A **firing** of an enabled transition removes one token from each input place and adds one token to each output place.

If a sequence of firings transforms a marking M into a marking M', we say that M' is **reachable** from M

A marked Petri net is **deadlocked** if no transition can fire.

A marking M for a Petri net is **live** if, beginning from M, no matter what sequence of firing occurs, it is possible to fire any given transition by proceeding through some additional firing sequence.

A marking M for a Petri net is **bounded** if there is some positive integer n such that in any firing sequence, no place ever receives more than n tokens.

If a marking M is bounded and in any firing sequence, no place ever receives more than one token, then M is a **safe** marking.